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A-Efficient Block Designs with Unequal Block Sizes for Comparing Two Sets of Treatments*

Seema Jaggi
IASRI, New Delhi

SUMMARY

This article studies an experimental setting in which it is desired to make comparisons between treatments belonging to two sets, each set consisting of two or more treatments. The experimental units are arranged in b blocks of varying sizes. Some general methods of construction of block designs for comparing two sets of treatments are given. Under this setting a sufficient condition is obtained to study the A-efficiency of these designs. A list of some of the designs obtained along with the parameters and A-efficiencies is also presented.

Key words : Balanced bipartite block designs; PBIB designs; BTIB designs; A-efficiency.

1. *Introduction*

A number of block designs are being used in agricultural experiments where the experimental situation requires one-way elimination of heterogeneity. In one of the experiments conducted at a Research Institute, 17 maize hybrids were required to be tested with 4 check (control) varieties of maize before making any recommendation for the 17 new hybrids. A similar type of situation arises when there are a number of equally popular or prevalent varieties under cultivation in the region, each having its own merit which can be used as controls for testing the new varieties, so that after the experiment is conducted one has enough freedom and scope in interpreting the performance of the varieties and identifying suitable varieties to be released for cultivation. It has been observed that such trials are generally being conducted in Randomized block design (RBD) with suitable number of replications. In the hybrid maize experiment also the varieties or treatments were laid out in a RBD with three replications.

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Using an RBD with 21 treatments is generally not desirable because it is not possible to form large homogeneous blocks to accomodate 21 treatments and the intra-block variances also become very large resulting thereby in a loss of efficiency. Among the incomplete block designs, a balanced incomplete block (BIB) design with parameters $v = b = 21$, $r = k = 5$, $\lambda = 1$ [for definition and parameters see e.g. Dey [6]] may be used instead of an RBD. For estimating all the $({}^{21}C_2)=210$ elementary contrasts, it is well known that a BIB design is most efficient. In fact, a BIB design, whenever existent, is universally optimal over D(v, b, k), where D (v, b, k) denotes the class of all connected designs with v treatments, b blocks and block size k. However, the interest of the experimenter is to make only the paired comparisons between 17 hybrid varieties of maize and 4 controls. In other words the interest is in making only 68 (17x4) paired comparisons. When the interest of the experimenter is in making paired comparisons on a subset of all the possible paired comparisons and when the other comparisons [e.g. the $({}^{17}C_2)=136$ paired comparisons between 17 hybrid varieties and $({}^{4}C_2)=6$ paired comparisons between control treatments] are not of any interest to the experimenter and do not have any role in the choice of the designs, then the designs useful for making all the possible paired comparisons efficiently do not remain so in the situation where only subset of paired comparisons are to be estimated.

Therefore, the BIB design does not remain optimal in this situation. Further, sometimes the nature of the experimental material may be such that it may not allow the blocks to be of equal sizes. For example when there are irregular fields or in hilly areas, the block designs having unequal block sizes can be adopted. The need for blocks of unequal sizes in biological experiments has been noted by Pearce [12]. This article makes an attempt to address this problem of obtaining efficient incomplete block designs with unequal block sizes for making paired comparisons between test treatments and control treatments when other comparisons between test treatments and control treatments are not of any interest to the experimenter and do not have any role in the choice of the design. The designs with unequal block sizes bring in the problem of heteroscedasticity in the model i.e. the intra-block variances become proportional to the block sizes. But here we consider the designs whose block sizes do not vary widely and also the number of block sizes are small. We would assume therefore that the intra-block variances are constant and the model is homoscedastic.

We now formulate the problem algebraically. Consider an experimental setting where $v = v_1 + v_2$ treatments are divided into two disjoint sets, one set T of cardinality v_1 called test treatments (herein after called tests) denoted by 1, 2, ..., v_1 and the other set S of cardinality v_2, called control treatments (herein after called controls) denoted by $v_1+1, \ldots, v_1+v_2$ such that $T \cap S = \varphi$ and these treatments are to be arranged in incomplete block design with b blocks and varying block sizes. All the comparisons of two treatments, one from T and one from S are important, but comparisons within T or within S are of less consequence i.e. the interest is in estimating the contrasts of the type $(\tau_t - \tau_s)$, $t \varepsilon T$, $s \varepsilon S$, which can be put in the form $\mathbf{P}'\tau$ where

$$P' = [1_{v_2} \otimes I_{v_1} : -I_{v_2} \otimes 1_{v_1}], \qquad \ldots(1.1)$$

is a $v_1v_2 \times v$ matrix and $\tau = [\tau_1, \ldots, \tau_{v_1}, \tau_{v_1+1}, \ldots, \tau_v]'$ is a $v \times 1$ vector of effects of tests and controls. 1_u is a ux1 vector of ones, I_u is an identity matrix of order u and $\otimes$ denotes the kronecker product of matrices.

A block design with b blocks of different block sizes will be used for the present setting. The model of response is given by

$$Y_{iju} = \mu + \tau_i + \beta_j + e_{iju} \qquad \ldots(1.2)$$

where Y_{iju} is the observation on treatment i in the plot u of block j, τ_i represents the effect of treatment i, β_j is the effect of block j, μ is the general mean and e_{iju}'s are uncorrelated random variables with mean zero and common variance σ^2. For a given block design, let $\mathbf{N} = ((n_{ij}))$ denote a vxb incidence matrix for i=1, ..., v; j=1, ..., b. N can also be written in partitioned form as

$$\mathbf{N} = [N_1 : N_2 : \ldots : N_p],$$

where N_l, $l = 1, \ldots, p$ $(\leq b)$ is the incidence matrix of the l^{th} part of the design i.e. the one with b_l blocks of size k_l, $\sum_{l=1}^{p} b_l = b$. The row sums of $\mathbf{N}$ are (r', r'_o), where $r'=(r_1, \ldots, r_{v_1})$ are the replication numbers of the tests and $r_o' = (r_{v_1+1}, \ldots, r_v)$ are the replication numbers of the controls. The column sums of N are $k' = (k_1 1'_{b_1}, \ldots, k_p 1'_{b_p})$, where k is the vector of block sizes and $\sum_{i=1}^{v} r_i = \sum_{l=1}^{p} b_l k_l = n$, the total number of observations.

Let $D(v_1, v_2, b, k_1, \ldots, k_p) = D$ denote the class of all connected designs in v_1 tests, v_2 controls arranged in b blocks such that b_l blocks are of size k_l, such that either all the controls appear together equally frequently in a block or do not appear at all. Also $D_o(v_1, v_2, b, k_1, \ldots, k_p) = D_o$ denotes the subclass of designs in D which are binary in tests. In order to study the A-efficiencies of the designs, the design is to be chosen from D which attains the minimum of $\sum_{t \in T} \sum_{s \in S} \mathrm{Var}(\hat{\tau}_{dt} - \hat{\tau}_{ds}) = \sigma^2 \text{ trace } (P' C_d^- P)$ as d varies over all D. Here $(\hat{\tau}_{dt} - \hat{\tau}_{ds})$ denotes the BLUE of $(\tau_{dt} - \tau_{ds})$ under the design d, C_d is the information matrix pertaining to the contrasts of interest and σ^2 is the error variance per plot. A design which attains this minimum is called an A-optimal design.

The purpose of the present article is to investigate the optimality aspects of designs for making paired comparisons between tests and controls. The concept of Balanced Bipartite Block (BBPB) designs of Kageyama and Sinha [9] has been extended and Balanced Bipartite block designs with unequal block sizes (BBPBUB) have been defined. In section 3 some general methods of construction of BBPBUB designs have been given. The results on A-optimality of the BBPBUB designs have been described in section 4 and A-optimality of some of the designs constructed is also investigated. For designs which are not A-optimal, A-efficiency has been worked out. A list of some of the BBPBUB designs giving A-efficiencies is also prepared.

2. *Some Preliminaries*

The problem of obtaining optimal incomplete block designs for making test treatments-control comparisons has been extensively studied by several researchers [see e.g. Hedayat, Jacroux and Majumdar [8] for an excellent review]. For the many controls situation which is the subject of discussion in the present article, Kageyama and Sinha [9] defined under the proper setting BBPB designs, extending the concept of Balanced treatment incomplete block (BTIB) designs for single control case. Kageyama and Sinha [9] and Sinha and Kageyama [13] have given some systematic methods of constructing BBPB designs together with tables of these designs. Majumdar [10] obtained sufficient conditions for a proper block design to be A-optimal for estimating differences of two treatments, one from each set, for the experimental situations with blocks of small size, assuming the design to be binary in tests as well as controls and under some assumptions of orthogonality for the experimental situations with large block sizes.

For comparing a set of tests to a single control in unequal blocks, the concept of Balanced treatment incomplete block designs with unequal block sizes (BTIUB) was given by Angelis and Moyssiadis [1]. Using the definition of BTIUB designs and BBPB designs, the definition of Balanced bipartite block designs with unequal block sizes (BBPBUB) have been introduced here.

Let $\lambda_{lii'}$ denote the number of times the pair of treatments $i \neq i' = 1, \ldots, v$ appear together in the block in the l^{th} part of the design. Let

$$f_{ts} = \sum_{l=1}^{p} \frac{\lambda_{lts}}{k_l}, \; t = 1, \ldots, v_1; \; s = v_1 + 1, \ldots, v$$

$$f_{tt'} = \sum_{l=1}^{p} \frac{\lambda_{ltt'}}{k_l}, \; t \neq t' = 1, \ldots, v_1$$

$$f_{ss'} = \sum_{l=1}^{p} \frac{\lambda_{lss'}}{k_l}, \; s \neq s' = v_1 + 1, \ldots, v$$

Definition 2.1. A block design is called a Balanced Bipartite Block design with unequal block sizes (BBPBUB) if

i) for every $t \in \{1, 2, \ldots, v_1\}$ and $s \in \{v_1 + 1, \ldots, v\}$, $f_{ts} = f_o$ for some constant f_o,

ii) for every $t \neq t' \in \{1, 2, \ldots, v_1\}$ $f_{tt'} = f_1$ for some constant f_1,

iii) for every $s \neq s' \in \{v_1 + 1, \ldots, v\}$, $f_{ss'} = f_2$ for some constant f_2.

The information matrix C of the design for estimating the treatment effects is given by

$$C = \begin{bmatrix} (a_1 - f_1) I_{v_1} + f_1 1_{v_1} 1'_{v_1} & f_o 1_{v_1} 1'_{v_2} \\ f_o 1_{v_2} 1'_{v_1} & (a_2 - f_2) I_{v_2} + f_2 1_{v_2} 1'_{v_2} \end{bmatrix} \qquad \ldots(2.1)$$

which is positive semi-definite with zero row (column) sums and a_1, a_2 are some scalar constants. It will be assumed throughout that the design is connected. The analysis of these designs is straight forward like any other general block design making use of the above C matrix. When p=b, all the b blocks of the design would be of distinct sizes, for p=1, a BBPBUB design as defined above is a BBPB design and for p=1, v_2=1, a BBPBUB design reduces to a BTIB design of Bechhofer and Tamhane [2].

3. *Methods of Construction of BBPBUB Designs*

The purpose of this section is to describe some methods of construction of BBPBUB designs supported with examples. The designs are obtained by using partially balanced incomplete block (PBIB) designs with two associate classes and cyclic designs [for definitions, see e.g. Dey [6]].

3.1 Using PBIB designs

This method extends the method of construction of variance balanced (VB) designs of Gupta and Jones [7] to obtain BBPBUB designs. The method is general in nature in the sense that one can have many distinct block sizes.

For l=1, ..., p let N_l be the incidence matrix of a PBIB design with two associate classes and with parameters $v_1, b_l, r_l, k_l, \lambda_{l1}, \lambda_{l2}$, m, n. Let v_2 be the number of controls to be added. If

$$\frac{\sum_{l=1}^{p} w_l \lambda_{l1}}{k_l + j_l v_2} = \frac{\sum_{l=1}^{p} w_l \lambda_{l2}}{k_l + j_l v_2}$$

where j_l is non-negative integer and w_l are the weights, then

$$N = \begin{bmatrix} 1'_{w_1} \otimes N_1 \ldots 1'_{w_l} \otimes N_l \ldots 1'_{w_p} \otimes N_p \\ j_1 1_{v_2} 1'_{w_1 b_1} \cdots j_l 1_{v_2} 1'_{w_l b_l} \cdots j_p 1_{v_2} 1'_{w_p b_p} \end{bmatrix} \qquad \ldots(3.1)$$

is the incidence matrix of a BBPBUB design with parameters $v_1^* = v_1$, $v_2^* = v_2$, $r^* = [(\sum_{l=1}^{p} w_l r_l) 1'_{v_1}, (\sum_{l=1}^{p} w_l j_l b_l) 1'_{v_2}]'$, $k^* = (k_1^{*\prime}, \ldots, k_p^{*\prime})'$,

$k_l^* = (k_l + j_l v_2) 1'_{b_l w_l}$, $b^* = \sum_{l=1}^{p} w_l b_l$. For illustration purposes we have studied only those designs for which $w_l = 1 \; \forall \; l = 1, \ldots, p$ and $j_l = 0$ or $1 \; \forall \; l = 1, \ldots, p$ and $p = 2$. The information matrix can be obtained easily using the above incidence matrix.

Remark 3.1.1 : If $k_l + j_l v_2 = k$ (say) for all l=1,..., p, then the method reduces to constructing proper BBPB designs.

Example 3.1.1. : Consider a Semi-regular Group Divisible (SRGD) design SR2 [Clatworthy [3]] with parameters $v_1 = 4$, $b_1 = 8$, $r_1 = 4$, $k_1 = 2$, m=2, n=2, $\lambda_{11} = 0$, $\lambda_{12} = 2$ and another regular GD design R1 with $v_1 = 4$, $b_2 = 8$, $r_2 = 4$, $k_2 = 2$, m = 2, n = 2, $\lambda_{21} = 2$, $\lambda_{22} = 1$. The design obtained by taking the blocks of these two designs together and augmenting two new treatments (v_2=2) in each block of the first design results in a BBPBUB design with parameters $v_1^* = 4$, $v_2^* = 2$, $r^* = 81'_6$, $k^* = [41'_8, 21'_8]'$, $b^* = 16$. The design is

(1, 2, 5, 6); (3, 4, 5, 6); (4, 1, 5, 6); (2, 3, 5, 6); (1, 2, 5, 6); (3, 4, 5, 6); (4, 1, 5, 6); (2, 3, 5, 6); (1, 3); (2, 4); (2, 4); (1, 3); (1, 2); (3, 4); (1, 4); (2, 3).

3.2 Using Cyclic Designs

Consider a cycle of α initial blocks such that $\alpha_l (>0)$ are of size k_l $(l = 1, \ldots, p)$ and $\sum_{l=1}^{p} \alpha_l = \alpha$. The α cycles when developed mod v_1 form a pairwise balanced design. Let $\lambda_{ltt'}$ be the frequency of the tt'^{th} pair of treatments in the l^{th} cycle of the design. Augmenting v_2 controls j_1 times in all the blocks of size k_1, j_2 times in the blocks of size k_2 and so on, j_p times in the blocks of size k_p, such that $\dfrac{\sum_{l=1}^{p} w_l \lambda_{ltt'}}{k_l + j_l v_2}$ = constant for all $t, t' = 1, \ldots, v_1$, results in a BBPBUB design with block sizes $k^* = (k_1^{*\prime}, \ldots, k_p^{*\prime})$, $k_l^* = (k_l + j_l v_2)$, where j_l is non-negative integer such that $\sum_{l=1}^{p} j_l \neq 0$ and w_l are the weights.

Remark 3.2.1 : Here if $k_l + j_l v_2 = k$ (say) for $l = 1, \ldots, p$, then the method reduces to constructing proper BBPB designs.

Example 3.2.1 : Consider the cyclic design with initial blocks (1, 2, 3, 5) and (1, 4) which are to be developed mod 8. Augmenting $v_2 = 4$ controls once in the blocks of size 2 i.e. $j_1 = 0$, $j_2 = 1$ and then taking two copies of design from first block i.e. w_1=2 and three copies of the design from second block i.e. w_2=3, results in a BBPBUB design with $v_1^* = 8$, $v_2^* = 4$, $r^* = [141'_8, 241'_4]'$ and $k^* = [41'_{16}, 61'_{24}]$.

4. *A-optimality of BBPBUB Designs*

This section is devoted to obtain a sufficient condition for a block design to be A-optimal for making paired comparisons between tests and controls. The interest here is to minimize the average variance of the BLUE of the parametric contrasts of interest, i.e. trace $(P' C_d^- P)$. Because of large variety of information matrices C_d among which the choice of an A-optimal design is to be made, the minimization problem becomes difficult to handle. We therefore make use of an intermediate device i.e. the averaged version M_d of C_d over all permutations $v_1!\,v_2!$.

Lemma 4.1 : The matrix M_d obtained by averaging information matrix C_d separately over treatments 1, ... , v_1 and v_1+1, ..., v is of the form

$$M_d = \begin{bmatrix} (\bar{a}_1 - \bar{f}_1) I_{v_1} + \bar{f}_1 1_{v_1} 1'_{v_1} & \bar{f}_o 1_{v_1} 1'_{v_2} \\ \bar{f}_o 1_{v_2} 1'_{v_1} & (\bar{a}_2 - \bar{f}_2) I_{v_2} + \bar{f}_2 1_{v_2} 1'_{v_2} \end{bmatrix} \qquad \ldots(4.1)$$

with

$$\bar{a}_1 = \frac{1}{v_1} \sum_{t=1}^{v_1} \sum_{l=1}^{p} [n_{tl} - k_l^{-1} n_{tl}^2];\ \bar{f}_1 = -\frac{1}{v_1(v_1-1)} \sum_{t \neq t'}^{v_1} \sum_{l=1}^{p} k_l^{-1} \lambda_{ltt'};$$

$$\bar{f}_o = -\frac{1}{v_1 v_2} \sum_{l=1}^{p} k_l^{-1} \sum_{t=1}^{v_1} \sum_{s=v_1+1}^{v} \lambda_{lts};\ \bar{a}_2 = \frac{1}{v_2} \sum_{s=v_1+1}^{v} \sum_{l=1}^{p} [n_{sl} - k_l^{-1} n_{sl}^2];$$

$$\bar{f}_2 = -\frac{1}{v_2(v_2-1)} \sum_{s \neq s'}^{v} \sum_{l=1}^{p} k_l^{-1} \lambda_{lss'}$$

Lemma 4.2 : [Majumdar, [10]]. Suppose Θ is a convex real valued possibly infinite function on the set of all non-negative matrices and Θ is invariant under permutations i.e. if $Q(\pi)$ be the permutation matrix, $\Theta[Q(\pi) C_d Q(\pi)'] = \Theta(C_d)$. Then for $d \in D$, $\Theta(C_d) \geq \Theta(M_d)$ and hence $\text{Trace}(P' C_d^- P) \geq \text{Trace}(P' M_d^- P)$.

Lemma 4.3 : Let $d \in D$ be a design which is not binary in test treatments. Then there exists $d^* \in D_o$ which is binary in test treatments with $r_{d*s} = r_{ds}$, for $s = v_1 + 1, \ldots, v$, where $r_{ds} = \sum_{j=1}^{b} n_{dsj}$ and which satisfies $\Theta(P' M_{d*}^- P) \leq \Theta(P' M_d^- P)$ and hence

$$\text{Trace}(P' M_{d*} P) \leq \text{Trace}(P' M_d^- P)$$

Lemma 4.4 : For $d \in D$,

$$\text{Trace}(P' M_d^- P) = \frac{v_2(v_1-1)}{\bar{a}_1 - \bar{f}_1} + \frac{v_1(v_2-1)}{\bar{a}_2 - \bar{f}_2} + \frac{v_2}{\bar{a}_1 + (v_1-1)\bar{f}_1} \qquad \ldots(4.2)$$

The minimization of this trace expression (4.2) becomes very difficult to handle as it involves minimization of a polynomial of order four. Therefore a condition is imposed that whenever any control treatment appears in a block, all the remaining (v_2 - 1) controls appear in that block equally frequently. Therefore for the class of designs considered here

$$\sum_{l=1}^{p} k_l^{-1} \sum_{s=v_1+1}^{v} n_{sl}^2 = v_2 \sum_{l=1}^{p} k_l^{-1} n_{ol}^2;$$

$$\sum_{s=v_1+1}^{v} \sum_{l=1}^{p} n_{sl} = v_2 \sum_{l=1}^{p} r_{ol};$$

$$\sum_{l=1}^{p} k_l^{-1} \sum_{s \neq s'=v_1+1}^{v} \lambda_{lss'} = v_2 (v_2 - 1) \sum_{l=1}^{p} k_l^{-1} n_{ol}^2$$

where $r_{ol} \geq 0$ is the replication number of the control treatment in the blocks of size k_l, 1 = 1, ..., p.

Lemma 4.5 : For a design $d \in D_o$ and fixed r_{ol}, Trace $(P' M_d^- P)$ is minimized when n_{ol} is either $[r_{ol}/b_l]$ or $[r_{ol}/b_l] + 1$.

Proof : For fixed r_{ol} and $d \in D_o$

$$\text{Trace}\,(P' M_d^- P) = \frac{v_1}{\sum_{l=1}^{p} r_{ol} - v_2 \sum_{l=1}^{p} k_l^{-1} n_{ol}^2} + \frac{v_1 (v_2 - 1)}{\sum_{l=1}^{p} r_{ol}}$$

$$+ \frac{v_1 v_2 (v_1 - 1)^2}{v_1 \sum_{l=1}^{p} \frac{(b_l k_l - v_2 r_{ol})(k_l - 1)}{k_l} - v_2 \sum_{l=1}^{p} r_{ol} + v_2^2 \sum_{l=1}^{p} k_l^{-1} n_{ol}^2} \qquad \text{...(4.3)}$$

Let $\sum_{l=1}^{p} k_l^{-1} n_{ol}^2 = Q$

Differentiating this expression with respect to Q and putting the result over a common denominator yields a ratio whose denominator is positive and whose numerator is a function of Q i.e.

$$h(Q) = -v_1 v_2^3 (v_1 - 1)^2 \{\sum_{l=1}^{p} r_{ol} - v_2 Q\}^2 + v_1 v_2$$

$$\left\{ v_1 \sum_{l=1}^{p} \frac{(b_l k_l - v_2 r_{ol})(k_l - 1)}{k_l} - v_2 \sum_{l=1}^{p} r_{ol} + v_2^2 Q \right\}^2$$

Coefficient of Q^2 in h(Q) is $-v_1^2 v_2^5 (v_1 - 2)$. For coefficient of Q^2 to be negative, $v_1 > 2$ which is true. Now

$$Q = \sum_{l=1}^{p} k_l^{-1} n_{ol}^2 \geq \sum_{l=1}^{p} k_l^{-1} r_{ol}^2/b_l = Q_1 \text{ (say)}$$

Since one of the positive eigen value of $P' M_d^- P$ is

$$v_1 / \left[\sum_{l=1}^{p} r_{ol} - v_2 \sum_{l=1}^{p} k_l^{-1} n_{ol}^2 \right] \geq 0$$

hence $Q \leq \sum_{l=1}^{p} r_{ol}/v_2 = Q_2$ (say)

For $Q_1 \leq Q \leq Q_2$, $h(Q_1) \geq 0$ and $h(Q_2) \geq 0$, it follows that $h(Q) \geq 0$ for all Q satisfying $Q_1 \leq Q \leq Q_2$. This implies that trace $(P' M_d^- P)$ is increasing in Q and is minimized for fixed r_{ol} when Q is as small as possible. For fixed r_{ol}, Q is minimized when n_{ol} is either $[r_{ol}/b_l]$ or $[r_{ol}/b_l] + 1$.

In view of the above lemmas, another definition of BBPBUB designs is given here which will be useful for searching A-optimal designs.

Definition 4.1 : For $w_l \in \{0, 1, \ldots, k_l - 1\}$ and $q_l \in \{0, 1, \ldots, b_l - 1\}$ for all $1 \leq l \leq p$ with atleast one $q_l > 0$ when $\sum_{l=1}^{p} w_l = 0$, a design d is a BBPBUB design with parameters v_1, v_2, b, $k_1, \ldots, k_p$; $w_1, \ldots, w_p$, $q_1, \ldots, q_p$ if it is a BBPBUB design with the additional property that

$$n_{tj} \in \{0, 1\},\ t = 1, \ldots, v_1;\ j = 1, \ldots, b \ (= \sum_{l=1}^{p} b_l)$$

$$n_{s1} = \dots = n_{sq_l} = w_l + 1, \; s = v_1 + 1, \dots, v$$

$$n_{s(q_l+1)} = \dots = n_{sb_l} = w_l \text{ for all } l = 1, \dots, p$$

We are now in a position to state the main theorem of this section which will be helpful in obtaining A-optimal designs.

Theorem 4.1 : Let $v_1, v_2, b, k_1, \dots, k_p$ be integers. A BBPBUB $D_o(v_1, v_2, b, k_1, \dots, k_p; w_1, \dots, w_p, q_1, \dots, q_p)$ design is A-optimal in the class of all designs with all the controls appearing together equally frequently in a block and with the same values of $v_1, v_2, b, k_1, \dots, k_p$ if

$$g(w_1, \dots, w_p, q_1, \dots, q_p) = \min \{g(x_1, \dots, x_p, z_1, \dots, z_p), \quad (x_l, z_l) \in \Delta\} \qquad \dots(4.4)$$

where $\Delta = \{ (x_l, z_l) \,.\, x_l = 0, 1, \dots, [k_l/v_2] - 1;$

$$z_l = 0, 1, \dots, b_l \text{ with } z_l > 0 \text{ when } \sum_{l=1}^{p} x_l = 0, 1 \leq l \leq p\},$$

$$g(x_1, \dots, x_p, z_1, \dots, z_p) = \frac{v_1 v_2 (v_1 - 1)^2}{A(x_1, \dots, x_p, z_1, \dots, z_p)} + \frac{v_1 (v_2 - 1)}{B(x_1, \dots, x_p, z_1, \dots, z_p)} + \frac{v_1}{C(x_1, \dots, x_p, z_1, \dots, z_p)} \qquad \dots(4.5)$$

$$A(x_1, \dots, x_p, z_1, \dots, z_p) = \sum_{l=1}^{p} [a_l (k_l - 1) - v_2 c_l + v_2^2 e_l]$$

$$B(x_1, \dots, x_p, z_1, \dots, z_p) = \sum_{l=1}^{p} c_l; \; C(x_1, \dots, x_p, z_1, \dots, z_p) = \sum_{l=1}^{p} [c_l - v_2 e_l]$$

with $a_l = v_1 \dfrac{(b_l k_l - v_2 b_l x_l - v_2 z_l)}{k_l}$; $c_l = b_l x_l + z_l$; $e_l = (b_l x_l^2 + 2x_l z_l + z_l)/k_l$

Proof : Using the definition 4.1 of BBPBUB design and $w_l = x_l$, $q_l = z_l$, we get $r_{ol} = b_l x_l + z_l$, $n_{ol}^2 = b_l x_l^2 + 2x_l z_l + z_l$. Using the expression (4.3), the required result is obtained.

A BBPBUB design $d_l \in D_o$ is a design in v_1 tests, v_2 controls arranged in blocks of size k_l each. The first q_l blocks in d_l referred to as d_{l1} have

$(k_l - w_l - 1)$ units each and the remaining $(b_l - q_l)$ blocks in d_l referred to as d_{2l} have $(k_l - w_l)$ units each in tests. Therefore it is easily seen that in both d_{1l} and d_{2l} the tests are equally replicated. Then

$$\sum_{t=1}^{v_1} n_{tl} = r_l,\ l \in d_l \text{ and } \sum_{t=1}^{v_1} n_{tl} = r_{1l},\ l \in d_{1l}$$

Now it is easy to establish the following lemma.

Lemma 4.6 : For a BBPBUB design $d \in D_o$, the following relations must hold

$$v_1 r_l = b_l (k_l - w_l) - q_l;\ v_1 r_{1l} = q_l\ (k_l - w_l - 1);$$

$$\lambda_{ltt'}\ (v_2 - 1) = r_{1l}\ (k_l - w_l - 2) + (r_l - r_{1l})\ (k_l - w_l - 1) \qquad ...(4.6)$$

Using Theorem 4.1, a number of A-optimal BBPBUB designs can be obtained. In search for an optimal design with parameters $v_1, v_2, b, k_1, \ldots, k_p$, we are led to the following algorithm.

Step 1 : Find integers $w_1, \ldots, w_p; q_1, \ldots, q_p$ which minimize the function $g(x_1, \ldots, x_p, z_1, \ldots, z_p)$ given in (4.4).

Step 2 : Verify the necessary conditions (4.6), i.e, check whether the following quantities are integers.

$$q_{1l} = [b_l (k_l - w_l) - q_l]/v_1$$

$$q_{2l} = q_l (k_l - w_l - 1)/v_1, \forall\ l = 1, \ldots, p$$

$$q_o (v_1 - 1) = \prod_{l=1}^{p} k_l \{ \sum_{l=1}^{p} [\, q_{2l}\ (k_l - w_l - 2) + (q_{1l} - q_{2l})\ (k_l - w_l - 1)]/k_l \}$$

Proceed if q_{1l}, q_{2l} and q_o are integers, otherwise theorem 4.1 cannot be applied.

5. *A-efficiency of BBPBUB Designs*

In section 4 a sufficient condition is obtained for studying the A- optimality of BBPBUB designs. It is however difficult to give general methods of construction which yields all the designs as A-optimal. Therefore, we look for an indirect approach and settle for a design that, though possibly not A-optimal, performs well under the A-criterion. The A-efficiencies of some of the BBPBUB designs constructed is computed using the definition of A-efficiency given by Stufken [14].

Definition 5.1 : The efficiency E(d) of a design $d \in D$ is defined as

$$E(d) = \frac{\min_{d^* \in D} \text{Trace}\,(P' M_{d*}^{-} P)}{\text{Trace}\,(P' M_d^{-} P)} \qquad \ldots (5.1)$$

where d^* is a hypothetical A-optimal design in D for which trace $(P' M_d^{-} P)$ is minimum. The designs with an efficiency exactly equal to one are termed as A-optimal. The A-efficiencies of some of the BBPBUB designs obtained through method 3.1 in section 3 is computed and the designs are listed in table 1 with $v_2^* = 2$ and average replications of tests and controls ≤ 15. From the table it is clear that most of the designs have high A-efficiency and some of the designs are also A-optimal with efficiency one. There may be some more designs with efficiency greater than those in the given class of designs, but here only those designs are reported which are obtained using the given method of construction.

Discussion : The designs discussed in the present article have been called by different names in literature. Nair and Rao [11] studied these designs under the name of an inter- and intra-group balanced design, Corsten [4] called them as balanced block designs with two different numbers of replicates etc. There is another class of designs called the General Efficiency Balanced (GEB) designs introduced by Das and Ghosh [5]. These designs can be used for comparing a set of tests to a set of controls. In fact all the GEB designs for two sets of treatments are BBPB designs but the converse may not be true always. The optimality problem attempted here involves minimization of average variance of the BLUE of contrasts of interest which is algebraically quite complicated, therefore the condition of all the controls appearing together in the block has been imposed. Efforts are to be made in solving this problem by relaxing this condition. The design for experiment mentioned in section 1 can be obtained by using cyclic design with initial blocks (1,3,6,10); (1,2); (1,7) developed mod 17 and taking $j_1 = 0$, $j_2 = j_3 = 1$, $p = 2$, $w_1 = 2$, $w_2 = w_3 = 3$.

Table 1. Some BBPBUB designs with two types of block sizes and $v_2^* = 2$

v_1^*	r_1^*	r_2^*	k_1^*	k_2^*	b_1^*	b_2^*	Reference design**		A-efficiency
4	8	8	4	2	8	8	SR2,	R1	1.0000
4	11	8	4	2	8	14	SR2,	R6	0.9859
4	14	8	4	2	8	20	SR2,	R16	0.9646
4	14	14	4	2	14	14	R7,	R6	0.9999
4	13	16	4	2	16	10	SR4,	R2	1.0000
4	10	10	4	2	10	10	R2,	R3	1.0000
4	11	14	4	2	14	8	R7,	R1	1.0000
4	13	10	4	2	10	16	R2,	R10	0.9896
6	6	6	6	3	6	4	S2,	SR18	0.9384
6	11	6	6	3	6	14	S2,	R46	0.5784
6	12	12	6	3	12	8	S4,	SR19	0.9384
8	6	6	6	3	6	8	S6,	R54	0.9197
8	12	12	6	3	12	16	S7,	R55	0.9196
8	14	16	6	2	16	24	R99,	R29	0.9899
8	12	16	6	4	16	8	R99,	SR36	0.9595
9	7	9	6	3	9	9	R104,	SR23	0.9611
9	10	18	6	3	18	6	LS27,	LS7	0.9963
9	12	12	8	4	12	9	LS73,	LS26	0.8832
10	10	10	6	3	10	20	T29,	T12	0.9027
10	9	15	6	2	15	15	T30,	T2	0.9999
10	10	10	8	4	10	10	T58,	T33	0.8751
12	8	8	8	4	8	12	SR66,	R109	0.8589
12	15	15	6	3	15	40	S11,	R77	0.8879
12	6	6	8	4	6	9	S27,	SR41	0.8591
12	12	12	8	4	12	18	S29,	SR42	0.8589
12	8	22	10	5	6	16	S54,	SR26	0.9995
12	14	14	8	4	14	21	R214,	R211	0.8589
12	8	8	8	4	8	12	SR66,	R109	0.8589
14	7	7	8	4	7	14	S32,	R112	0.8449
14	14	14	8	4	14	28	S33,	R113	0.8449
18	14	15	8	6	15	27	S38,	SR73	0.8148
18	15	15	8	4	15	45	S38,	R222	0.8216

** Source of PBIB designs is Clatworthy [3]

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